TRIANGULATIONS OF NON-PROPER SEMIALGEBRAIC THOM MAPS

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Abstract. In [5] I solved the Thom's conjecture that a proper Thom map is triangulable. In this paper I drop the properness condition in the semialgebraic case and, moreover, in the definable case in an o-minimal structure.

1. Introduction

Let r be always a positive integer or ∞, X and Y subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and $f : X \to Y$ a C^r map (i.e., f is extended to a C^r map from an open neighborhood of X in $\mathbf{R}^m$ to one of Y in $\mathbf{R}^n$). A C^r *stratification* of f is a pair of C^r stratifications $\{X_i\}$ of X and $\{Y_j\}$ of Y such that for each i, the image $f(X_i)$ is included in some Y_j and the restriction map $f|_{X_i} : X_i \to Y_j$ is a C^r submersion. We call also $f : \{X_i\} \to \{Y_j\}$ a C^r *stratification* of $f : X \to Y$. We call $f : X \to Y$ a *Thom C^r map* if there exists a Whitney C^r stratification $f : \{X_i\} \to \{Y_j\}$ such that the following condition is satisfied. Let X_i and $X_{i'}$ be strata with $X_{i'} \cap (\overline{X}_i - X_i) = \emptyset$. If $\{a_k\}$ is a sequence of points in X_i converging to a point b of $X_{i'}$ and if the sequence of the tangent spaces $\{T_{a_k}(f|_{X_i})^{-1}(f(a_k))\}$ converges to a space $T \subset \mathbf{R}^m$ in the Grassmannien space $G_{m,m'}$, $m' = \dim(f|_{X_i})^{-1}(f(a_k))$, then $T_b(f|_{X_{i'}})^{-1}(f(b)) \subset T$. We call then $f : \{X_i\} \to \{Y_j\}$ a *Thom C^r stratification* of $f : X \to Y$. In [5] I solved the following Thom's conjecture.

Theorem 1.1. *Assume X and Y are closed in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and $f : X \to Y$ is a proper Thom C^∞ map. Then there exist homeomorphisms τ and π from X and Y to polyhedra P and Q, respectively, such that $\pi \circ f \circ \tau^{-1} : P \to Q$ is piecewise-linear.*

Here a natural question arises. Whether can we drop the properness condition? Indeed, the condition is too strong for some applications. For example, the natural map from a G-manifold M to its orbit space is a Thom map but not necessarily proper provided the action $G \times M \ni (g, x) \to (gx, x) \in M^2$ is proper (see [2]). In the present paper we give a positive answer in the semialgebraic or definable case. A C^r stratification $f : \{X_i\} \to \{Y_j\}$ of $f : X \to Y$ is called *semialgebraic* (*definable*) if X, Y, f, X_i and Y_j are all semialgebraic (definable, respectively,) and $\{X_i\}$ and $\{Y_j\}$ are finite stratifications.

Theorem 1.2. *Assume X and Y are closed and semialgebraic (definable in an o-minimal structure) in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and $f : X \to Y$ is a semialgebraic*

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(definable, respectively,) Thom C^1 map. Then there exist finite simplicial complexes K and L and semialgebraic (definable, respectively,) C^0 imbeddings $\tau : X \to |K|$ and $\pi : Y \to |L|$ such that $\tau(X)$ and $\pi(Y)$ are unions of some open simplexes of K and L, respectively, and $\pi \circ f \circ \tau^{-1} : \tau(X) \to \pi(Y)$ is extended to a simplicial map from K to L, where $|K|$ denotes the underlying polyhedron to K.

The theorem does not necessarily hold without the condition that X is closed in $\mathbf{R}^m$. A counter-example is given by $X = \mathbf{R}^2 - \{(x, y) \in \mathbf{R}^2 : x = 0,\ y \neq 0\},\ Y = \mathbf{R}^2$ and $f(x, y) = (x, xy)$. Such f is not triangulable in the weak sense that there exist C^0 imbeddings τ of X and π of Y into some Euclidean space $\mathbf{R}^n$ such that $\overline{\tau(X)}$ is a polyhedron and $\pi \circ f \circ \tau^{-1} : \tau(Y) \to \pi(X)$ is extended to a piecewise-linear map $\theta : \overline{\tau(X)} \to \mathbf{R}^n$ for the following reason. Assume there exist τ and π as required. Then $\overline{\tau(X)}$ is of dimension two and $\theta^{-1}(y)$ is of dimension 0 for each $y \in \overline{\pi(Y)}$ because θ is piecewise-linear and $\theta|_{\tau(X)}$ is injective. Hence a small compact neighborhood U of $\tau(0)$ in $\overline{\tau(X)}$ does not intersect with $\theta^{-1}(\pi(0))$ except at $\tau(0)$. Choose a point (x_1, x_2) in X with $x_2 \neq 0$ so close to 0 that the half-open segment L with ends $(0, x_2)$ and (x_1, x_2) in X is included in $\tau^{-1}(U)$. Then $\overline{f(L)} - f(L) = \{0\}$ and $\overline{\pi \circ f(L)} - \pi \circ f(L) = \{\pi(0)\}$. Hence $(\overline{\tau(L)} - \tau(L)) \cap U = \{\tau(0)\}$ or $(\overline{\tau(L)} - \tau(L)) \cap U = \emptyset$ since $\theta^{-1}(\pi(0)) = \{\tau(0)\}$ in U. The former case contradicts the definition of L and the fact that τ is a C^0 imbedding, and the latter does the fact that U is compact.

An open problem is whether a Thom C^1 map $f : X \to Y$ is triangulable in this weak sense under the condition that X is closed in $\mathbf{R}^n$ or, equivalently, X is locally compact.

2. Tube systems

If r is larger than one, C^r *tube* at a C^r submanifold M of $\mathbf{R}^n$ is a triple $T = (|T|, \pi, \rho)$, where $|T|$ is an open neighborhood of M in $\mathbf{R}^n$, $\pi : |T| \to M$ is a submersive C^r retraction and ρ is a non-negative C^r function on $|T|$ such that $\rho^{-1}(0) = M$ and each point x on M is a unique and non-degenerate critical point of $\rho|_{\pi^{-1}(x)}$. We will need to consider a C^1 tube. Assume M is a C^1 submanifold of $\mathbf{R}^n$. Let $|T|$ be an open neighborhood of M in $\mathbf{R}^n$, $\pi : |T| \to M$ a C^1 map and ρ a C^1 function on $|T|$. We call $T = (|T|, \pi, \rho)$ a C^1 *tube* at M if there exists a C^1 imbedding τ of $|T|$ into $\mathbf{R}^n$ such that $\tau(M)$ is a C^2 submanifold of $\mathbf{R}^n$ and $\tau_* T = (\tau(|T|), \tau \circ \pi \circ \tau^{-1}, \rho \circ \tau^{-1})$ is a C^2 tube at $\tau(M)$. (See pages 33–40 in [4], which says the arguments on tube systems in [1] work in the C^1 category.) A C^r *tube system* $\{T_j\}$ for a C^r stratification $\{Y_j\}$ of a set $Y \subset \mathbf{R}^n$ consists of one tube T_j at each Y_j. We define a C^r *weak tube system* $\{T_j = (|T_j|, \pi_j, \rho_j)\}$ for the same $\{Y_j\}$ weakening the conditions on ρ_j as follows. Each ρ_j is a non-negative C^0 function on $|T_j|$ with zero set Y_j, of class C^r on $|T_j| - Y_j$ and regular on $Y_{j'} \cap \pi_j^{-1}(y) - Y_j$ for each $y \in Y_j$ and $Y_{j'}$. Note a C^r tube system is a C^r weak tube system if $\{Y_j\}$ is a Whitney stratification by Lemma I.1.1, [4]. In the following arguments we shrink $|T_j|$ many times without mention.

We call a C^r (weak) tube system $\{T_j\}$ for $\{Y_j\}$ *controlled* if for each pair j and j' with $(\overline{Y_{j'}} - Y_{j'}) \cap Y_j \neq \emptyset$,

$$\pi_j \circ \pi_{j'} = \pi_j \quad \text{and} \quad \rho_j \circ \pi_{j'} = \rho_j \quad \text{on } |T_j| \cap |T_{j'}|.$$

Remember there exists a controlled C^r tube system for a Whitney stratification (see [1] and [4]), note if $\{T_j\}$ is such a C^r tube system then the map $(\pi_j, \rho_j)|_{Y_{j'}\cap|T_j|}$ is a C^r submersion into $Y_j \times \mathbf{R}$ because

$$(\pi_j, \rho_j)|_{Y_{j'}\cap|T_j|} \circ \pi_{j'} = (\pi_j, \rho_j) \quad \text{on } |T_j| \cap |T_{j'}|,$$

and if we assume only $\pi_j \circ \pi_{j'} = \pi_j$ on $|T_j| \cap |T_{j'}|$ then $\pi_j|_{Y_{j'}\cap|T_j|}$ is a C^r submersion into Y_j. In the case of a C^r weak tube system $(\pi_j, \rho_j)|_{Y_{j'}\cap|T_j|-Y_j}$ is a C^1 submersion into $Y_j \times \mathbf{R}$. Let $f : \{X_i\} \to \{Y_j\}$ be a C^r stratification of a C^r map $f : X \to Y$ between subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, $\{T_j^Y = (|Y_j^Y|, \pi_j^Y, \rho_j^Y)\}$ a controlled C^r (weak) tube system for $\{Y_j\}$ and $\{T_i^X = (|T_i^X|, \pi_i^X, \rho_i^X)\}$ a C^r (weak) tube system for $\{X_i\}$. We call $\{T_i^X\}$ *controlled over* $\{T_j^Y\}$ if the following four conditions are satisfied. Let f be extended to a C^r map $\tilde{f} : \cup_i|T_i^X| \to \mathbf{R}^n$.

(1) For each (i, j) with $f(X_i) \subset Y_j$,

$$f \circ \pi_i^X = \pi_j^Y \circ \tilde{f} \quad \text{on } |T_j^X| \cap \tilde{f}^{-1}(|T_j^Y|).$$

(2) For each j, $\{T_i^X : f(X_i) \subset Y_j\}$ is a controlled C^r (weak) tube system for $\{X_i : f(X_i) \subset Y_j\}$.

(3) For each pair i and i' with $(\overline{X_{i'}} - X_{i'}) \cap X_i \neq \emptyset$,

$$\pi_i^X \circ \pi_{i'}^X = \pi_i^X \quad \text{on } |T_i^X| \cap |T_{i'}^X|.$$

(4) For each (i, j) with $f(X_i) \subset Y_j$ and (i', j') with $(\overline{X_{i'}} - X_{i'}) \cap X_i \neq \emptyset$ and $f(X_{i'}) \subset Y_{j'}$, $(\pi_i^X, f)|_{X_{i'}\cap|T_i^X|}$ is a C^r submersion into the fiber product $X_i \times_{(f,\pi_j^Y)} (Y_{j'} \cap |T_j^Y|)$—the C^r manifold $\{(x, y) \in X_i \times (Y_{j'} \cap |T_j^Y|) : f(x) = \pi_j^Y(y)\}$.

Note (4) is equivalent to the next condition.

(4)′ For (i, j), (i', j') as in (4) and for each $x \in X_{i'} \cap |T_i^X|$, the germ of $\pi_i^X|_{X_{i'}\cap f^{-1}(f(x))}$ at x is a C^r submersion onto the germ of $X_i \cap f^{-1}(\pi_j^Y \circ f(x))$ at $\pi_i^X(x)$.

This definition of controlledness is stronger than that in [1]. In [1], (4) is not assumed. However, if $f : \{X_i\} \to \{Y_j\}$ is a Thom map then (4) immediately follows from (1), (2) and (3), and existence of a C^r tube system $\{T_i^X\}$ for $\{X_i\}$ controlled over a given controlled C^r tube system $\{T_j^Y\}$ for $\{Y_j\}$ is known (see [1] and [4]). We shall treat a C^1 stratification $f : \{X_i\} \to \{Y_j\}$ of f which is not necessarily a Thom C^1 stratification but admits a controlled C^1 tube system $\{T_j^Y\}$ for $\{Y_j\}$ and a C^1 weak tube system $\{T_i^X\}$ for $\{X_i\}$ controlled over $\{T_j^Y\}$.

In [5] theorem 1.1 is proved in the following more general form.

Theorem 2.1. *Let $f : \{X_i\} \to \{Y_j\}$ be a C^∞ stratification of a C^∞ proper map $f : X \to Y$ between closed subsets of Euclidean spaces. Assume there exist a controlled C^∞ tube system $\{T_j^Y\}$ for $\{Y_j\}$ and a C^∞ tube system $\{T_i^X\}$ for $\{X_i\}$ controlled over $\{T_j^Y\}$. Then there exist homeomorphisms τ and π from X and Y to polyhedra P and Q, respectively, closed in some Euclidean spaces such that $\pi \circ f \circ \tau^{-1} : P \to Q$ is piecewise linear and $\tau(\overline{X_i})$ and $\pi(\overline{Y_j})$ are all polyhedra. If $f : \{X_i\} \to \{Y_j\}$, $\{T_i^X\}$ and $\{T_j^Y\}$ are semialgebraic or, more generally, definable in an o-minimal structure, then we can choose semialgebraic or definable τ, π, P and Q.*

(Note a semialgebraic closed polyhedron in a Euclidean space is semilinear, i.e., is defined by a finite number of equalities and inequalities of linear functions.) Moreover, the proof in [5] shows the following generalization though we do not repeat its proof.

Theorem 2.2. *Let $f : \{X_i\} \to \{Y_j\}$ be a C^1 stratification of a C^1 proper map $f : X \to Y$ between closed subsets of Euclidean spaces. Let I denote the set of indexes i of X_i such that $f|_{X_i}$ is not injective. Assume there exist a controlled C^1 tube system $\{T_j^Y\}$ for $\{Y_j\}$ and a C^1 weak tube system $\{T_i^X\}$ for $\{X_i\}$ controlled over $\{T_j^Y\}$ such that $\{T_i^X : i \in I\}$ is a C^1 tube system for $\{X_i : i \in I\}$. Then the result in theorem 2.1 holds.*

We will prove theorem 1.2 by compactifying $f : X \to Y$ in theorem 1.2 and applying theorem 2.2 to the compctification. There are two unusual problems which we encounter. First the arguments do not work in the C^2 category and apply the C^1 category. Secondly we construct $\{T_j^Y : Y_j \subset \overline{Y}\}$ and $\{T_i^X : X_i \subset X\}$ by induction on $\dim Y_j$ and $\dim X_i$ but the induction of construction of $\{T_i^X : X_i \subset \overline{X} - X\}$ is downward. The two inductions are not independent and we need special conditions (iv) and (ix) for tube systems in the proof below. It is natural to ask whether we can extend f to a Thom map $\overline{f}$. The answer is negative. To keep the property that f is a Thom map also we use (iv) and (ix).

3. Proof theorem 1.2

Proof of theorem 1.2. We assume X is non-compact and X and Y are bounded in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, by replacing $\mathbf{R}^m$ and $\mathbf{R}^n$ with $(0,\,1)^m$ and $(0,\,1)^n$ respectively. Then $\overline{X}-X$ and $\overline{Y}-Y$ are compact. Let $f : \{X_i\} \to \{Y_j\}$ be a semialgebraic Thom C^1 stratification of $f : X \to Y$. Then we can assume f is extendable to $\overline{X}$. Apply Theorem II.4.1, [3] to the function on $\mathbf{R}^m$ measuring distance from the compact set $\overline{X}-X$. Then we have a non-negative semialgebraic C^0 function ϕ on $\mathbf{R}^m$ such that $\phi^{-1}(0) = \overline{X} - X$ and $\phi|_{\mathbf{R}^m - (\overline{X}-X)}$ is of class C^1. Choose $\epsilon > 0 \in \mathbf{R}$ so that ϕ is C^1 regular on $\phi^{-1}((0,\,\epsilon])$ and let ϕ' be a semialgebraic C^1 function on $\mathbf{R}$ such that $\phi'(0) = 0$, ϕ' is regular on $(0,\,\epsilon)$ and $\phi' = 1$ on $[\epsilon,\,\infty)$. Set

$$\Phi(x) = (\phi' \circ \phi(x), \phi' \circ \phi(x)x) \quad \text{for } x \in X.$$

Then Φ is a semialgebraic C^1 imbedding of X into $\mathbf{R}^{m+1}$ such that $\Phi(X)$ is bounded and $\overline{\Phi(X)} - \Phi(X) = \{0\}$. Hence replacing X with $\Phi(X)$ we assume $\overline{X} - X = \{0\}$ from the beginning. Moreover, replace X with the graph of f. Then we suppose X is contained and bounded in $\mathbf{R}^m \times \mathbf{R}^n$, $\overline{X} - X \subset \{0\} \times \overline{Y}$, $f : X \to Y$ is the restriction of the projection $p : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ and hence f is extended to a semialgebraic C^1 map $\overline{f} : \overline{X} \to \overline{Y}$.

By the same reason we assume $\overline{Y} - \{0\}$. Note then $\{Y_j, 0\}$ is a semialgebraic Whitney C^1 stratification of $\overline{Y}$. Let $\{T_j^Y\}$ be a controlled semialgebraic C^1 tube system for $\{Y_j\}$ and $\{T_i^X\}$ a semialgebraic C^1 tube system for $\{X_i\}$ controlled over $\{T_j^Y\}$. Assume the set of indexes of Y_j does not contain 0, set $Y_0 = \{0\}$ and add Y_0 to $\{Y_j\}$. Then we can assume there is a semialgebraic C^1 tube $T_0^Y = (|T_0^Y|, \pi_0^Y, \rho_0^Y)$ at Y_0 such that $\{T_j^Y, T_0^Y : j \neq 0\}$ is controlled for the following reason.

Let $|T_0^Y|$ be the closed ball $B(\epsilon)$ with center 0 in $\mathbf{R}^n$ and with small radius $\epsilon > 0$ (we treat closed balls in place of open balls for simplicity of notation), and set $\pi_0^Y(y) = 0$ and, tentatively, $\rho_0^Y(y) = |y|^2$ for $y \in |T_0^Y|$. Then the condition $\rho_0^Y \circ \pi_j^Y = \rho_0^Y$ on $|T_0^Y| \cap |T_j^Y|$ for $j \neq 0$ does not necessarily hold. For that condition it suffices to find a semialgebraic homeomorphism τ of $\mathbf{R}^n$ of class C^1 outside of 0 and such that $\tau(0) = 0$, $\tau = \mathrm{id}$ outside of $B(\epsilon)$ and $\rho_0^Y \circ \pi_j^Y \circ \tau^{-1} = \rho_0^Y$ on $B(\epsilon') \cap \tau(|T_j^Y|)$ for $j \neq 0$, shrunk $|T_j^Y|$ and some $\epsilon' > 0$.

Let Y_j be such that $\dim Y_j$ is the smallest in $\{Y_j : 0 \in \overline{Y_j},\ j \neq 0\}$, and choose ϵ so small that $\rho_0^Y|_{Y_j \cap |T_0^Y|}$ is C^1 regular, which implies that $\rho_0^{Y-1}(\epsilon'^2)$ is transversal to Y_j for any $0 < \epsilon' \leq \epsilon$. Set $Y_j(\epsilon') = Y_j \cap \rho_0^{Y-1}(\epsilon'^2)$. We will define a semialgebraic homeomorphism τ_j of $\mathbf{R}^n$ of class C^1 outside of 0 such that $\tau_j(0) = 0$, $\tau_j = \mathrm{id}$ outside of $B(\epsilon)$ and $\rho_0^Y \circ \pi_j^Y \circ \tau_j^{-1} = \rho_0^Y$ on $B(\epsilon/2) \cap \tau_j(|T_j^Y|)$ for shrunk $|T_j^Y|$. Since the problem is local at Y_j, we can assume by Thom's first isotopy lemma (see Theorem II.6.1 and it complement, [4]) that

$$|T_0^Y| \cap Y_j = Y_j(\epsilon) \times (0,\ \epsilon^2],\ \text{ after then, } |T_0^Y| \cap |T_j^Y| = \cup\{y + L_y : y \in Y_j(\epsilon)\} \times (0,\ \epsilon^2]$$

and $\pi_j^Y(y+z,t)$ and $\rho_0^Y(y+z,t)$ are of the form $(y, \pi_j^{Y\prime}(y+z,t))$ and t, respectively, for $y \in Y_j(\epsilon)$ and $(z,t) \in L_y \times (0,\ \epsilon^2]$, where L_y is a linear subspace of the tangent space $T_y\rho_0^{Y-1}(\epsilon^2)$ of codimension $=$ codim Y_j in $\mathbf{R}^n$ such that the correspondence $Y_j(\epsilon) \ni y \to L_y \in G_{n,\mathrm{codim}\, Y_j}$ is semialgebraic and of class C^1 and $\pi_j^{Y\prime}$ is a semialgebraic C^1 function defined on $\cup\{y + L_y\} \times (0,\ \epsilon^2]$. For simplicity of notation we write $\cup_{y \in Y_j(\epsilon)}\{y\} \times L_y$ as $Y_j(\epsilon) \times L$. Transform $Y_j(\epsilon) \times L \times (0,\ \epsilon^2]$ by a semialgebraic C^1 diffeomorphism $(y,z,t) \to (y, z/kt^k, t)$ for sufficiently large integer k. Then we can assume

$$(0) \qquad |\pi_j^{Y\prime}(y+z,t) - t| \leq \epsilon^2/28 \ \text{ and } \ |\frac{\partial \pi_j^{Y\prime}}{\partial t}(y+z,t) - 1| < 1/4 \ \text{ for } |z| \leq 1$$

since
$$\pi_j^{Y\prime}(y,t) = t.$$

Let ξ be a semialgebraic C^1 function on $\mathbf{R}$ such that $0 \leq \xi \leq 1$, $\xi = 1$ on $(-\infty,\ 1/2)$, $\xi = 0$ on $(2/3,\ \infty)$ and $|\frac{d\xi}{dt}| \leq 7$. Set

$$\tau_j(y+z,t) = (y+z, (1 - \xi(2t/\epsilon^2)\xi(|z|))t + \xi(2t/\epsilon^2)\xi(|z|)\pi_j^{Y\prime}(y+z,t))$$
$$\text{for } (y,z,t) \in Y_j(\epsilon) \times L \times (0,\ \epsilon^2].$$

Then $\tau_j = \pi_j^Y$ if $t \leq \epsilon^2/4$ and $|z| \leq 1/2$, $\tau_j = \mathrm{id}$ if $t \geq \epsilon^2/3$ or $|z| \geq 2/3$ and, moreover, τ_j is a diffeomorphism because

$$|\frac{\partial}{\partial t}\big((1 - \xi(t/\epsilon^2)\xi(|z|))t + \xi(t/\epsilon^2)\xi(|z|)\pi_j^{Y\prime}(y+z,t)\big) - 1|$$
$$\leq \xi(t/\epsilon^2)\xi(|z|)|1 - \frac{\partial \pi_j^{Y\prime}}{\partial t}(y+z,t)| + |\frac{d\xi}{dt}(t/\epsilon^2)\xi(|z|)|t - \pi_j^{Y\prime}(y+z,t)|/\epsilon^2$$
$$\leq 1/4 + 1/4 = 1/2 \quad \text{for } |z| \leq 1.$$

Thus we can assume $\rho_0^Y \circ \pi_j^Y = \rho_0^Y$ on $|T_0^Y| \cap |T_j^Y|$.

Repeating the same arguments by induction on $\dim Y_{j'}$ for all $Y_{j'}$ with $0 \in \overline{Y_{j'}}$ we obtain the required τ. Here we note only that for j' with $\overline{Y_{j'}} - Y_{j'} \supset Y_j$, though $Y_{j'}(\epsilon)$ is not compact, (0) can holds. Indeed

$$\rho_0^Y = \rho_0^Y \circ \pi_j^Y \circ \pi_{j'}^Y = \rho_0^Y \circ \pi_{j'}^Y \quad \text{on } |T_0^Y| \cap |T_j^Y| \cap |T_{j'}^Y|.$$

Hence when we describe $\pi_{j'}^Y$ as above there is a semialgebraic neighborhood U of $Y_j(\epsilon) \times (0,\, \epsilon^2]$ in $\overline{Y_{j'}(\epsilon)} \times (0,\, \epsilon^2]$ such that

$$\pi_{j'}^{Y\prime}(y+z,t) = t \quad \text{for } (y,z,t) \in Y_{j'}(\epsilon) \times L_y \times (0,\, \epsilon^2] \text{ with } (y,t) \in U.$$

In conclusion we assume Y is compact.

If $f : \{X_i\} \to \{Y_j\}$ is extended to a Thom C^1 stratification of $\overline{f} : \overline{X} \to Y$, then theorem 1.2 follows from theorem 1.1 in the C^1 case. However, such extension does not always exist. Instead we will find a semialgebraic C^1 stratification $\overline{f} : \{X'_{i'}\} \to \{Y'_{j'}\}$ of $\overline{f}$, a controlled semialgebraic C^1 tube system $\{T_{j'}^{Y\prime}\}$ for $\{Y'_{j'}\}$ and a semialgebraic C^1 weak tube system $\{T_{i'}^{X\prime}\}$ for $\{X'_{i'}\}$ controlled over $\{T_{j'}^{Y\prime}\}$ such that $\{X'_{i'}\}|_X$ and $\{Y'_{j'}\}|_Y$ are substratifications of $\{X_i\}$ and $\{Y_j\}$. Here $\{Y'_{j'}\}$ is a Whitney stratification but $\{X'_{i'}\}$ is not necessarily so.

Set $Z = \overline{X} - X$, which is compact. Note $Z = \{0\} \times \overline{f}(Z)$ and $\overline{f}|_Z$ is a homeomorphism onto $\overline{f}(Z)$. Let $\{Y'_{j'}\}$ be a semialgebraic Whitney C^1 substratification of $\{Y_j\}$ such that each stratum is connected, $\overline{f}(Z)$ is a union of some $Y'_{j'}$'s and $\{X_i, \{0\} \times (Y'_{j'} \cap \overline{f}(Z))\}$ is a Whitney C^1 stratification of $\overline{X}$, which is constructed in the same way as the canonical semialgebraic C^ω stratification of a semialgebraic set since $\overline{f}(Z)$ is closed in Y. Note $\{Y'_{j'}\}$ satisfies the frontier condition. Set

$$\{X'_{i'}\} = \{X_i \cap \overline{f}^{-1}(Y'_{j'}), Z \cap \{0\} \times Y'_{j'}\}.$$

Then $\{X'_{i'}\}$ is a semialgebraic (not necessarily Whitney) C^1 stratification of $\overline{X}$; $\{X'_{i'} \cap X\}$ is a substratification of $\{X_i\}$; $\overline{f} : \{X'_{i'}\} \to \{Y'_{j'}\}$ is a C^1 stratification of $\overline{f}$; we can choose $\{Y'_{j'}\}$ so that for each $Y'_{j'}$, $\{X'_{i'} : \overline{f}(X'_{i'}) = Y'_{j'}\}$ is a Whitney C^1 stratification for the following reason.

Assume $Y'_{j'} \not\subset \overline{f}(Z)$. Then $Y'_{j'} \cap \overline{f}(Z) = \emptyset$ and there is Y_j including $Y'_{j'}$. By definition of $\{X'_{i'}\}$,

$$\{X'_{i'} : \overline{f}(X'_{i'}) = Y'_{j'}\} = \{X_i \cap f^{-1}(Y'_{j'})\}.$$

Therefore the assertion follows from the fact that given a Whitney C^r stratification $\{M_1, M_2\}$, a C^r map g from $M_1 \cup M_2$ to a C^r manifold N such that $g|_{M_1}$ and $g|_{M_2}$ are C^r submersions into N and a C^r submanifold N_1 of N then $\{M_1 \cap g^{-1}(N_1), M_2 \cap g^{-1}(N_1)\}$ is a Whitney C^r stratification.

Next assume $Y'_{j'} \subset \overline{f}(Z)$, and let $X'_{i'_1}$ and $X'_{i'_2}$ be such that $\overline{f}(X'_{i'_k}) = Y'_{j'},\ k = 1,2$, and $(\overline{X'_{i'_1}} - X'_{i'_1}) \cap X'_{i'_2} \neq \emptyset$. Then we need to see $(X'_{i'_1}, X'_{i'_2})$ can satisfy the Whitney condition. Since $\overline{f}|_Z$ is injective, there are only two possible cases to consider: $X'_{i'_k} = X_{i_k} \cap \overline{f}^{-1}(Y'_{j'}),\ k = 1,2$, for some i_1 and i_2 or $X'_{i'_1} = X_{i_1} \cap \overline{f}^{-1}(Y'_{j'})$ and $X'_{i'_2} = \{0\} \times Y'_{j'}$. In the former case there is j such that $Y'_{j'} \subset Y_j$. Hence the Whitney condition is

satisfied by the same reason as in the case of $Y'_{j'} \not\subset \overline{f}(Z)$. Consider the latter case. If $\{X'_{i'_1}, \{0\}\times Y'_{j'}\}$ is not a Whitney stratification, let $Y''_{j'}$ denote the subset of $Y'_{j'}$ consisting of y such that $(X'_{i_1}, \{0\}\times Y'_{j'})$ does not satisfy the Whitney condition at $(0,y)$. Then $Y''_{j'}$ and hence $\overline{Y''_{j'}}$ are semialgebraic and of dimension smaller that $\dim Y'_{j'}$. Divide $Y'_{j'}$ to $\{Y'_{j'} - \overline{Y''_{j'}}, \overline{Y''_{j'}}\}$ and substratify $\{Y'_{j'}\cap \overline{f}(Z)\}$ by downward induction on dimension of $Y'_{j'}$ so that the above conditions on $\{Y'_{j'}\}$ are kept and $Y''_{j'} = \emptyset$. Then $\{X'_{i'_1}, \{0\}\times Y'_{j'}\}$ becomes a Whitney stratification.

Now we define a controlled semialgebraic C^1 tube system $\{T^{Y\prime}_{j'} = (|T^{Y\prime}_{j'}|, \pi^{Y\prime}_{j'}, \rho^{Y\prime}_{j'})\}$ for $\{Y'_{j'}\}$. For simplicity of notation, assume $\dim Y_j = j$ gathering strata of the same dimension. For each j, set

$$J_j = \left\{ \begin{array}{ll} \{j' : Y'_{j'} \subset Y_j,\} & \text{if } j \geq 0, \\ \emptyset & \text{if } j = -1. \end{array} \right.$$

We define $\{T^{Y\prime}_{j'} : j' \in J_j\}$ by induction on j. Fix a non-negative integer j_0, and assume we have constructed a controlled semialgebraic C^1 tube system $\{T^{Y\prime}_{j'} : j' \in J_j, j < j_0\}$ so that $T^{Y\prime}_{j'} = T^Y_{j_1}|_{|T^{Y\prime}_{j'}|}$ for $j' \in J_{j_1},\ j_1 < j_0$, with $\dim Y'_{j'} = j_1$,

$$(*)_Y \qquad \pi^{Y\prime}_{j'} \circ \pi^Y_j = \pi^{Y\prime}_{j'} \quad \text{on } |T^{Y\prime}_{j'}| \cap |T^Y_j| \text{ for } j' \text{ and } j \text{ with } Y'_{j'} \subset \overline{Y_j},$$

$$(**)_Y \qquad \rho^{Y\prime}_{j'} \circ \pi^Y_j = \rho^{Y\prime}_{j'} \quad \text{on } |T^{Y\prime}_{j'}| \cap |T^Y_j| \text{ for } j' \in J_{j_1} \text{ and } j \text{ with } j_1 < j,$$

$\pi^{Y\prime}_{j'}$ are of class C^1 and $\rho^{Y\prime}_{j'}$ are of class C^1 on $|T^{Y\prime}_{j'}| - Y'_{j'}$. For the conditions of the first and $(**)_Y$ we need to proceed in the C^1 category because there does not necessarily exist such $\{T^{Y\prime}_{j'}\}$ of class C^2 even if $\{T^Y_j\}$ is of class C^2.

We wil define a semialgebraic C^1 tube system $\{T^{Y\prime}_{j'} : j' \in J_{j_0}\}$ for $\{Y'_{j'} : j' \in J_{j_0}\}$. For the time being, let $\{T^{Y\prime}_{j'} : j' \in J_{j_0}\}$ be a semialgebraic C^1 tube system for $\{Y'_{j'} : j' \in J_{j_0}\}$ such that $\{T^{Y\prime}_{j'} : j' \in J_j, j \leq j_0\}$ is controlled (Lemma II.6.10, [4] states only the case where $\cup_{j'\in J_{j_0}} Y'_{j'}$ is compact but its proof works in the general case. We omit the details.) We modify $\{T^{Y\prime}_{j'} : j' \in J_{j_0}\}$ so that the conditions are satisfied. Let $j' \in J_{j_0}$.

Restrict $\pi^{Y\prime}_{j'}$ and $\rho^{Y\prime}_{j'}$ to Y_{j_0} for $j' \in J_{j_0}$ and define afresh them outside of Y_{j_0} as follows. Let $\pi^{Y\prime}_{j'}$ and $\rho^{Y\prime}_{j'}$, $j' \in J_{j_0}$, now denote the restrictions. If $\dim Y'_{j'} = j_0$, we should set $T^{Y\prime}_{j'} = T^Y_{j_0}|_{|T^{Y\prime}_{j'}|}$. Then $(*)_Y$ and $(**)_Y$ are satisfied because $\{Y^Y_j\}$ is controlled. Assume $\dim Y'_{j'} < j_0$ and hence $j_0 > 0$. In this case, define the extension of $\pi^{Y\prime}_{j'}$ to $|T^{Y\prime}_{j'}|$ to be $\pi^{Y\prime}_{j'} \circ \pi^Y_{j_0}$, and keep the same notation $\pi^{Y\prime}_{j'}$ for the extension. Then by controlledness of $\{T^Y_j\}$, $(*)_Y$ holds for any j with $Y'_{j'} \subset \overline{Y_J}$. The problem is how to extend $\rho^{Y\prime}_{j'}$.

As the problem is local at $Y'_{j'}$ (see II.1.1, [4]), considering semialgebraic tubular neighborhoods of $Y'_{j'}$ and Y_{j_0} we can assume for each $y \in Y'_{j'}$, $\pi^{Y\prime-1}_{j'}(y)$, $\pi^{Y\prime-1}_{j'}(y) \cap Y_{j_0}$ and $\pi^{Y-1}_{j_0}(y)$ are of the form $y + L_y$, $y + L_{0,y}$ and $y + L^{\perp}_{0,y}$, where L_y and $L_{0,y}$ are linear subspaces of $\mathbf{R}^n$ with $L_y \supset L_{0,y}$ and $L^{\perp}_{0,y}$ is the orthocomplement of $L_{0,y}$ with respect to L_y, and $\pi^Y_{j_0}|_{\pi^{Y\prime-1}_{j'}(y)} : \pi^{Y\prime-1}_{j'}(y) \longrightarrow \pi^{Y\prime-1}_{j'}(y)\cap Y_{j_0}$ is induced by the orthogonal projection

of L_y to $L_{0,y}$ and

$$\rho^Y_{j_0}(y+z_1+z_2)=|z_2|^2 \text{ for } (y,z_1,z_2)\in Y'_{j'}\times L_{0,y}\times L^{\perp}_{0,y},$$

where $Y'_{j'}\times L_{0,y}\times L^{\perp}_{0,y}$ denotes $\cup_{y\in Y'_{j'}}\{y\}\times L_{0,y}\times L^{\perp}_{0,y}$.

Set $\qquad \rho^{Y}_{j'}{}''(y+z_1+z_2)=|z_1|^2+|z_2|^2 \quad \text{for } (y,z_1,z_2)\in Y'_{j'}\times L_{0,y}\times L^{\perp}_{0,y}.$

Then $(|T^{Y}_{j'}{}'|,\pi^{Y}_{j'}{}',\rho^{Y}_{j'}{}'')$ is a semialgebraic C^1 tube at $Y'_{j'}$ but not always satisfy the condition $\rho^{Y}_{j'}{}''\circ\pi^Y_{j_0}=\rho^{Y}_{j'}{}''$. We need to modify $\rho^{Y}_{j'}{}''$ so that the equality holds on a neighborhood of $Y_{j_0}-Y'_{j'}$. Let ξ be a semialgebraic C^1 function on $\mathbf{R}$ such that $\xi=1$ on $(-\infty,1]$, $\xi=0$ on $[2,\infty)$ and $d\xi/dt\le 0$. Set

$$\eta_{j'}(z_1,z_2)=\begin{cases}\xi(\frac{|z_2|}{|z_1|^2})\frac{|z_1|}{(|z_1|^2+|z_2|^2)^{1/2}}+1-\xi(\frac{|z_2|}{|z_1|^2}) & \text{for } (z_1,z_2)\in(L_{0,y}-\{0\})\times L^{\perp}_{0,y},\\ 1 & \text{for } (z_1,z_2)\in\{0\}\times L^{\perp}_{0,y},\end{cases}$$

and define a semialgebraic map $\tau_{j'}$ between $|T^{Y}_{j'}{}'|$ by

$$\tau_{j'}(y+z_1+z_2)=y+\eta_{j'}(z_1,z_2)z_1+\eta_{j'}(z_1,z_2)z_2 \quad \text{for } (y,z_1,z_2)\in Y'_{j'}\times L_{0,y}\times L^{\perp}_{0,y}.$$

Then $\pi^{Y}_{j'}{}'\circ\tau_{j'}=\pi^{Y}_{j'}{}'$;

$$\tau_{j'}=\mathrm{id}\quad \text{on } \{y+z_1+z_2:|z_2|\ge 2|z_1|^2\};$$

$$\tau_{j'}(y+z_1+z_2)=y+\frac{|z_1|}{(|z_1|^2+|z_2|^2)^{1/2}}z_1+\frac{|z_1|}{(|z_1|^2+|z_2|^2)^{1/2}}z_2$$
$$\text{for } (y,z_1,z_2)\in Y'_{j'}\times L_{0,y}\times L^{\perp}_{0,y} \text{ with } |z_2|\le|z_1|^2;$$

$(**\,*)_Y \qquad \rho^{Y}_{j'}{}''\circ\tau_{j'}(y+z_1+z_2)=|z_1|^2 \quad$ for the same (y,z_1,z_2);

for each line l in $\{y\}\times L_{0,y}\times L^{\perp}_{0,y}$ passing through 0 parameterized by $t\in\mathbf{R}$ as $z_1=z_1(t)$ and $z_2=z_2(t)$ so that $|z_1(t)|=|t|$ and $|z_2(t)|=a|t|$ for $a\ge 0\in\mathbf{R}$,

$$\tau_{j'}(l)=l,$$

$$|\tau_{j'}(y+z_1(t)+z_2(t))-y|=\eta_{j'}(z_1(t),z_2(t))(|z_1(t)|^2+|z_2(t)|^2)^{1/2}$$
$$=\xi(\frac{a}{|t|})|t|+(1-\xi(\frac{a}{|t|}))(1+a^2)^{1/2}|t|,$$

hence by easy calculations we see if a is sufficiently small then $\tau_{j'}|_l$ is a C^1 diffeomorphism of l and, therefore by the above equality $\tau_{j'}=\mathrm{id}$ on $\{|z_2|\ge 2|z_1|^2\}$ shrinking $|T^{Y}_{j'}{}'|$ we can assume $\tau_{j'}$ is a homeomorphism and its restriction to $|T^{Y}_{j'}{}'|-Y'_{j'}$ is a C^1 diffeomorphism; moreover, if we set $\rho^{Y}_{j'}{}'=\rho^{Y}_{j'}{}''\circ\tau_{j'}$ and $T^{Y}_{j'}{}'=(|T^{Y}_{j'}{}'|,\pi^{Y}_{j'}{}',\rho^{Y}_{j'}{}')$ for all $j'\in J_{j_0}$ with $\dim Y'_{j'}<j_0$ then $\{T^{Y}_{j'_1}{}':j'_1\in J_{j_1},j_1\le j_0\}$ is a controlled semialgebraic C^1 tube system. Indeed, for $j'_1\in J_{j_0}$ and j'_2 with $(\overline{Y'_{j'_1}}-Y'_{j'_1})\cap Y'_{j'_2}\ne\emptyset$, the following equalities folds on $|T^{Y}_{j'_1}{}'|\cap|T^{Y}_{j'_2}{}'|$

$$\pi^{Y}_{j'_2}{}'\circ\pi^{Y}_{j'_1}{}'=\pi^{Y}_{j'_2}{}'\circ\pi^{Y}_{j'_1}{}'\circ\pi^Y_{j_0}\quad \text{by definition of } \pi^{Y}_{j'_1}{}'$$
$$=\pi^{Y}_{j'_2}{}'\circ\pi^Y_{j_0}\quad \text{by controlledness of } \{T^{Y}_{j'}{}'|_{Y_{j_0}}:j'\in J_j,\ j\le j_0\}$$
$$=\pi^{Y}_{j'_2}{}' \text{ by definition of } \pi^{Y}_{j'_2}{}' \text{ in the case of } j'_2\in J_{j_0} \text{ and by } (*)_Y \text{ in the other case.}$$

In the same way we see by $(**)_Y$ and $(***)_Y$

$$\rho^{Y\prime}_{j'_2}\circ\pi^{Y\prime}_{j'_1}=\rho^{Y\prime}_{j'_2}\quad\text{on } |T^{Y\prime}_{j'_1}|\cap|T^{Y\prime}_{j'_2}|.$$

Hence it remains to show $\tau_{j'}$ is a C^1 diffeomorphism.

It is easy to show $\tau_{j'}$ is differentiable at $Y'_{j'}$ and its differential $d\tau_{j'a}$ at each point a of $Y'_{j'}$ is equal to the identity map. Hence we only need to show the map $|T^{Y\prime}_{j'}|\ni a\to d\tau_{j'a}\in GL(\mathbf{R}^n)$ is of class C^0. As the problem is local at each point of $Y'_{j'}$ we suppose

$$Y'_{j'}=\mathbf{R}^{n'}\times\{0\}\times\{0\},\ Y_{j_0}=\mathbf{R}^{n'}\times\mathbf{R}^{n_1}\times\{0\},\ |T^{Y\prime}_{j'}|=|T^Y_{j_0}|=\mathbf{R}^{n'}\times\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$$

and $\pi^Y_{j_0}$ and $\pi^{Y\prime}_{j'}$ are the projections of $\mathbf{R}^{n'}\times\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ to $\mathbf{R}^{n'}\times\mathbf{R}^{n_1}\times\{0\}$ and $\mathbf{R}^{n'}\times\{0\}\times\{0\}$ respectively. Then it suffices to see the differential at (z_{01},z_{02}) of the map $\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}\ni(z_1,z_2)\to(\eta_{j'}(z_1,z_2)z_1,\eta_{j'}(z_1,z_2)z_2)\in\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ converges to the identity map as $(z_{01},z_{02})\to(0,0)$. That is,

$$d\left(\frac{\xi(\frac{|z_2|}{|z_1|^2})((|z_1|^2+|z_2|^2)^{1/2}-|z_1|)z_i}{(|z_1|^2+|z_2|^2)^{1/2}}\right)_{(z_{01},z_{02})}=$$

$$d\left(\frac{\xi(\frac{|z_2|}{|z_1|^2})|z_2|^2z_i}{(|z_1|^2+|z_2|^2)^{1/2}((|z_1|^2+|z_2|^2)^{1/2}+|z_1|)}\right)_{(z_{01},z_{02})}\longrightarrow 0$$

as $(z_{01},z_{02})\to(0,0)$ with $|z_2|\le 2|z_1|^2,\ i=1,2$, since $\eta_{j'}(z_1,z_2)=1$ for (z_1,z_2) with $|z_2|\ge 2|z_1|^2$. That is easy to check. We omit the details.

Thus we obtain semialgebraic C^1 tubes $T^{Y\prime}_{j'}$ for all $j'\in J_{j_0}$. The other requirements in the induction hypothesis are satisfied as follows. By definition of $T^{Y\prime}_{j'}$,

$$T^{Y\prime}_{j'}=T^Y_{j_0}|_{|T^{Y\prime}_{j'}|}\quad\text{for } j'\in J_{j_0}\text{ with }\dim Y'_{j'}=j_0;$$

by controlledness of $\{T^Y_j\}$ and by definition of $T^{Y\prime}_{j'}$, for j' and j with $Y'_{j'}\subset\overline{Y_j},\ j'\in J_{j_0}$ and $j\ge j_0$,

$$(*)_Y\qquad \pi^{Y\prime}_{j'}\circ\pi^Y_j=\pi^{Y\prime}_{j'}\circ\pi^Y_{j_0}\circ\pi^Y_j=\pi^{Y\prime}_{j'}\circ\pi^Y_{j_0}=\pi^{Y\prime}_{j'}\quad\text{on } |T^{Y\prime}_{j'}|\cap|T^Y_j|;$$

$(**)_Y$ holds for j' and j with $j'\in J_{j_0}$ and $j>j_0$ for the following reason.

That is clear if $\dim Y'_{j'}=j_0$. Hence assume $\dim Y'_{j'}<j_0$ and use the above coordinate system $Y'_{j'}\times L_{0,y}\times L^\perp_{0,y}$. Then

$$\rho^{Y\prime}_{j'}(y+z_1+z_2)=\rho^{Y\prime\prime}_{j'}\circ\tau_{j'}(y+z_1+z_2)=\eta^2_{j'}(z_1,z_2)(|z_1|^2+|z_2|^2)$$
$$\text{for } (y,z_1,z_2)\in Y'_{j'}\times L_{0,y}\times L^\perp_{0,y}$$

and $\eta_{j'}(z_1,z_2)$ depends on only $|z_1|$ and $|z_2|$. Hence if we set

$$\pi^Y_j(y+z_1+z_2)=\pi^Y_{j1}(y+z_1+z_2)+\pi^Y_{j2}(y+z_1+z_2)+\pi^Y_{j3}(y+z_1+z_2),$$

$$\pi^Y_{j1}(y+z_1+z_2)\in Y'_{j'},\ \pi^Y_{j2}(y+z_1+z_2)\in L_{0,y},\ \pi^Y_{j3}(y+z_1+z_2)\in L^\perp_{0,y}.$$

then it suffices to see

$$\pi^Y_{j2}(y+z_1+z_2)=z_1\quad\text{and}\quad|\pi^Y_{j3}(y+z_1+z_2)|=|z_2|.$$

By controlledness of $\{T^Y_j\}$ we have $\pi^Y_{j_0}\circ\pi^Y_j=\pi^Y_{j_0}$. Hence by the equation $\pi^Y_{j_0}(y+z_1+z_2)=y+z_1$, the former equality holds. The latter also follows from the equations $\rho^Y_{j_0}\circ\pi^Y_j=\rho^Y_{j_0}$ and $\rho^Y_{j_0}(y+z_1+z_2)=|z_2|^2$.

Hence by induction we have a controlled semialgebraic C^1 tube system $\{T^{Y\prime}_{j'}\}$ for $\{Y'_{j'}\}$ such that $T^{Y\prime}_{j'}=T^Y_j|_{|T^{Y\prime}_{j'}|}$ for $j'\in J_j$ with $\dim Y'_{j'}=j$, $(*)_Y$ for j' and j with $Y'_{j'}\subset\overline{Y_j}$ and $(**)_Y$ for $j'\in J_{j_1}$ and j with $j_1<j$.

Next we define $\{T^{X\prime}_{i'}\}$ by induction as $\{T^{Y\prime}_{j'}\}$. Consider all $X'_{i'}$ included in X and forget $X'_{i'}$ outside of X. We change the set of indexes of X_i. For non-negative integers i_0 and j_0, let X_{i_0,j_0} denote the union of X_i's such that $\dim X_i=i_0$ and $f(X_i)\subset Y_{j_0}$, i.e., $\dim f(X_i)=j_0$, naturally define $T^X_{i,j}=(|T^X_{i,j}|,\pi^X_{i,j},\rho^X_{i,j})$ and continue to define $\{X'_{i'}\}$ to be $\{X_{i,j}\cap p^{-1}(Y'_{j'}), Z\cap\{0\}\times Y'_{j'}\}$. Then $\dim X_{i,j}=i$ and $f|_{X_{i,j}}$ is a map to Y_j. Let I_i denote the set of indexes of $X'_{i'}$ such that $X'_{i'}$ is included in $X_{i,j}$ for some j. Note $X=\cup\{X'_{i'}: i'\in I_i \text{ for some } i\}$. Fix a non-negative integer i_0, and assume there exists a semialgebraic C^1 tube system $\{T^{X\prime}_{i'}=(|T^{X\prime}_{i'}|,\pi^{X\prime}_{i'},\rho^{X\prime}_{i'}) : i'\in I_i,\ i<i_0\}$ for $\{X'_{i'}: i'\in I_i,\ i<i_0\}$ such that the following four conditions are satisfied, which are, except (iv), similar to the conditions (1), (2) and (3) in section 2.

(i) For $i,\ i'$ and j' with $i<i_0,\ i'\in I_i$ and $f(X'_{i'})=Y'_{j'}$,

$$f\circ\pi^{X\prime}_{i'}=\pi^{Y\prime}_{j'}\circ p\quad\text{on } |T^{X\prime}_{i'}|\cap p^{-1}(|T^{Y\prime}_{j'}|).$$

(ii) For each j', $\{T^{X\prime}_{i'}: f(X'_{i'})=Y'_{j'},\ i'\in I_i,\ i<i_0\}$ is a controlled semialgebraic C^1 tube system for $\{X'_{i'}: f(X'_{i'})=Y'_{j'},\ i'\in I_i,\ i<i_0\}$.

(iii) For $i_k,\ i'_k,\ k=1,2,3,\ i_4$ and j_4 with $i_k<i_0,\ i'_k\in I_{i_k},\ k=1,2,3,\ X'_{i'_1}\cap(\overline{X'_{i'_2}}-X'_{i'_2})\neq\emptyset$ and $X'_{i'_3}\subset\overline{X_{i_4,j_4}}$,

$$\pi^{X\prime}_{i'_1}\circ\pi^{X\prime}_{i'_2}=\pi^{X\prime}_{i'_1}\quad\text{on } |T^{X\prime}_{i'_1}|\cap|T^{X\prime}_{i'_2}|,$$

$$\pi^{X\prime}_{i'_3}\circ\pi^{X}_{i_4,j_4}=\pi^{X\prime}_{i'_3}\quad\text{on } |T^{X\prime}_{i'_3}|\cap|T^{X}_{i_4,j_4}|,$$

if $i_3<i_4$ moreover, then

$$\rho^{X\prime}_{i'_3}\circ\pi^{X}_{i_4,j_4}=\rho^{X\prime}_{i'_3}\quad\text{on } |T^{X\prime}_{i'_3}|\cap|T^{X}_{i_4,j_4}|.$$

(iv) For $i,\ i'$ and j with $i<i_0,\ i'\in I_i$ and $\dim X'_{i'}=i$,

$$T^{X\prime}_{i'}=T^X_{i,j}|_{|T^{X\prime}_{i'}|}.$$

Then we need to define $\{T^{X\prime}_{i'}: i'\in I_{i_0}\}$ so that the induction process works. Before that we note a fact.

(v) Given $i_k,\ i'_k,\ j'_k,\ k=1,2$, with $i_k<i_0,\ i'_k\in I_{i_k},\ k=1,2,\ X'_{i'_1}\cap(\overline{X'_{i'_2}}-X'_{i'_2})\neq\emptyset$, $Y'_{j'_1}\subset\overline{Y'_{j'_2}}-Y'_{j'_2}$ and $f(X'_{i'_k})=Y'_{j'_k},\ k=1,2$, then the restriction of the map $(\pi^{X\prime}_{i'_1},f)$ to $X'_{i'_2}\cap|T^{X\prime}_{i'_1}|$ is a C^1 submersion into the fiber product $X'_{i'_1}\times_{(f,\pi^{Y\prime}_{j'_1})}(Y'_{j'_2}\cap|T^{Y\prime}_{j'_1}|)$.

The reason is the following.

Case where $X'_{i'_k}\subset X_{i_k,j_k},\ k=1,2$, for some $j_1\neq j_2$. The condition (4) in section 2 is shown to be equivalent to $(4)'$. Now also similar equivalence holds. Hence it suffices to

see for each $x \in X'_{i'_2} \cap |T^{X'}_{i'_1}|$, the germ of $\pi^{X'}_{i'_1}|_{X'_{i'_2} \cap f^{-1}(f(x))}$ at x is a C^1 submersion onto the germ of $X'_{i'_1} \cap f^{-1}(\pi^{Y'}_{j'_1} \circ f(x))$ at $\pi^{X'}_{i'_1}(x)$. We have four properties.

$$X'_{i'_2} \cap f^{-1}(f(x)) = X_{i_2,j_2} \cap f^{-1}(f(x)) \quad \text{by definition of } \{X'_{i'}\};$$

$$\begin{aligned} X'_{i'_1} \cap f^{-1}(\pi^{Y'}_{j'_1} \circ f(x)) &= X'_{i'_1} \cap f^{-1}(f \circ \pi^{X'}_{i'_1}(x)) \quad \text{by (i)} \\ &= X_{i_1,j_1} \cap f^{-1}(f \circ \pi^{X'}_{i'_1}(x)) \quad \text{by definition of } \{X'_{i'}\}; \end{aligned}$$

by $(4)'$ the germ of $\pi^X_{i_1,j_1}|_{X_{i_2,j_2} \cap f^{-1}(f(x))}$ at x is a C^1 submersion onto the germ of $X_{i_1,j_1} \cap f^{-1}(f \circ \pi^X_{i_1,j_1}(x))$ at $\pi^X_{i_1,j_1}(x)$; by (iii)

$$\pi^{X'}_{i'_1} \circ \pi^X_{i_1,j_1} = \pi^{X'}_{i'_1} \quad \text{on } |T^{X'}_{i'_1}| \cap |T^X_{i_1,j_1}|.$$

Hence we only need to see the germ of $\pi^{X'}_{i'_1}|_{X_{i_1,j_1} \cap f^{-1}(f \circ \pi^X_{i_1,j_1}(x))}$ at $\pi^X_{i_1,j_1}(x)$ is a C^1 submersion onto the germ of $X'_{i'_1} \cap f^{-1}(f \circ \pi^{X'}_{i'_1}(x))$ at $\pi^{X'}_{i'_1}(x)$. That is clear by (i) because $f|_{X_{i_1,j_1}} : X_{i_1,j_1} \to Y_{j_1}$ is a C^1 submersion onto a union of some connected components of Y_{j_1} and $f \circ \pi^X_{i_1,j_1}(x)$ and $f \circ \pi^{X'}_{i'_1}(x)$ are contained in the same connected component.

Note we use the hypothesis $X'_{i'_k} \subset X_{i_k,j_k},\ k = 1,2,\ j_1 \neq j_2$ in the above arguments for only the property that the germ of $\pi^X_{i_1,j_1}|_{X_{i_2,j_2} \cap f^{-1}(f(x))}$ is a C^1 submersion into $X_{i_1,j_1} \cap f^{-1}(f \circ \pi^X_{i_1,j_1}(x))$.

Case where $i_1 \neq i_2$ and $X'_{i'_k} \subset X_{i_k,j_k},\ k = 1,2$, for some j_1. In this case also the above property holds because $f \circ \pi^X_{i_1,j_1} = f$ on $X_{i_2,j_1} \cap |T^X_{i_1,j_1}|$ and $\pi^X_{i_1,j_1}|_{X_{i_2,j_1} \cap |T^X_{i_1,j_1}|}$ is a C^1 submersion into X_{i_1,j_1}.

Case where $i_1 = i_2$ and hence $X'_{i'_k} \subset X_{i_1,j_1},\ k = 1,2$, for some j_1. In this case the reason is simply $\pi^X_{i_1.j_1}|_{X_{i_1,j_1}} = \mathrm{id}$.

Thus (v) is proved. Now we define $\{T^{X'}_{i'} : i' \in I_{i_0}\}$. For that it suffices to consider separately $\{X'_{i'} : X'_{i'} \subset X_{i_0,j}\}$ for each j. Hence we assume all $X'_{i'}$ with $i' \in I_{i_0}$ are included in one $X_{i_0.j_0}$ for some j_0 and, moreover, $f(X_{i_0,j_0}) = Y_{j_0}$ for simplicity of notation. Then as shown below we have a semialgebraic C^1 tube system $\{T^{X'}_{i'} = (|T^{X'}_{i'}|, \pi^{X'}_{i'}, \rho^{X'}_{i'}) : i' \in I_0\}$ for $\{X'_{i'} : i' \in I_0\}$ such that
(vi) for i' and j' with $i' \in I_{i_0}$ and $f(X'_{i'}) = Y'_{j'}$,

$$f \circ \pi^{X'}_{i'} = \pi^{Y'}_{j'} \circ p \quad \text{on } |T^{X'}_{i'}| \cap p^{-1}(|T^{Y'}_{j'}|);$$

(vii) for $j' \in J_{j_0}$, $\{T^{X'}_{i'} : f(X'_{i'}) = Y'_{j'},\ i' \in I_{i_1},\ i_1 \leq i_0\}$ is a controlled semialgebraic C^1 tube system for $\{X'_{i'} : f(X'_{i'}) = Y'_{j'},\ i' \in I_{i_1},\ i_1 \leq i_0\}$;
(viii) for i_1, i'_k, $k = 1,2,3$, i_4 and j_4 with $i_1 \leq i_0$, $i'_1 \in I_{i_1}$, $i'_2, i'_3 \in I_{i_0}$, $X'_{i'_1} \cap (\overline{X'_{i'_2}} - X'_{i_2}) \neq \emptyset$ and $X'_{i'_3} \subset \overline{X_{i_4,j_4}}$,

$$\begin{aligned} \pi^{X'}_{i'_1} \circ \pi^{X'}_{i'_2} &= \pi^{X'}_{i'_1} \quad \text{on } |T^{X'}_{i'_1}| \cap |T^{X'}_{i'_2}|, \\ \pi^{X'}_{i'_3} \circ \pi^X_{i_4,j_4} &= \pi^{X'}_{i'_3} \quad \text{on } |T^{X'}_{i'_3}| \cap |T^X_{i_4,j_4}|, \end{aligned}$$

if $i_0 < i_4$ then

$$\rho^{X'}_{i'_3} \circ \pi^X_{i_4,j_4} = \rho^{X'}_{i'_3} \quad \text{on } |T^{X'}_{i'_3}| \cap |T^X_{i_4,j_4}|;$$

(ix) for $i' \in I_{i_0}$ with $\dim X'_{i'} = i_0$,

$$T^{X'}_{i'} = T^X_{i_0,j_0}|_{|T^{X'}_{i'}|}.$$

We construct $\{T^{X'}_{i'} : i' \in I_0\}$ as follows. First we define $T^{X'}_{i'}$ on $|T^{X'}_{i'}| \cap X_{i_0,j_0}$, $i' \in I_{i_0}$, so that (vi), (vii) and the first equality in (viii) are satisfied by the usual arguments of lift of a tube system (see [1], Lemma II.6.1, [4] and its proof). Secondly, extend $\pi^{X'}_{i'}$ to $|T^{X'}_{i'}|$ using $\pi^X_{i_0,j_0}$ as in the above construction of $\pi^{Y'}_{j'}$. Then $\pi^{X'}_{i'}$ are of class C^1; (vi) holds because for i' and j' with $i' \in I_{i_0}$ and $f(X'_{i'}) = Y'_{j'}$,

$$\begin{aligned} f \circ \pi^{X'}_{i'} &\overset{\text{definition of } \pi^{X'}_{i'}}{=} f \circ \pi^{X'}_{i'} \circ \pi^X_{i_0,j_0} \overset{\text{(vi) on } |T^{X'}_{i'}| \cap X_{i_0,j_0}}{=} \pi^{Y'}_{j'} \circ f \circ \pi^X_{i_0,j_0} \\ &\overset{\text{(1) in section 2}}{=} \pi^{Y'}_{j'} \circ \pi^Y_{j_0} \circ p \overset{(**)_Y}{=} \pi^{Y'}_{j'} \circ p \quad \text{on } |T^{X'}_{i'}| \cap p^{-1}(|T^{Y'}_{j'}|); \end{aligned}$$

the first equality in (viii) for $i_1 = i_0$ follows from definition of the extension; that for $i_1 < i_0$ does from the second equality in (iii); the second in (viii) does from definition of the extension and the equality $\pi^X_{i_0,j_0} \circ \pi^X_{i_4,j_4} = \pi^X_{i_0,j_0}$; trivially $\pi^{X'}_{i'} = \pi^X_{i_0,j_0}$ for $i' \in I_{i_0}$ with $\dim X'_{i'} = i_0$. Thirdly, extend $\rho^{X'}_{i'}$ to $|T^{X'}_{i'}|$ in the same way as $\rho^{Y'}_{j'}$. Then $\{T^{X'}_{i'} : i' \in I_{i_0}\}$ is a semialgebraic C^1 tube system for $\{X'_{i'} : i' \in I_{i_0}\}$; (vii) holds because for i'_0 and $i'_1 \in I_{i_1}$ with $i'_0 \in I_{i_0}$, $i_1 < i_0$ and $f(X'_{i'_0}) = f(X'_{i'_1})$,

$$\begin{aligned} \rho^{X'}_{i'_1} \circ \pi^{X'}_{i'_0} &= \rho^{X'}_{i'_1} \circ \pi^{X'}_{i'_0} \circ \pi^X_{i_0,j_0} \quad \text{by definition of } \pi^{X'}_{i'_0} \\ &= \rho^{X'}_{i'_1} \circ \pi^X_{i_0,j_0} \quad \text{by (vii) on } X_{i_0,j_0} \\ &= \rho^{X'}_{i'_1} \quad \text{by the third equality in (iii);} \end{aligned}$$

the extensions are chosen so that the third equality in (viii) and (ix) are satisfied, which completes construction of a semialgebraic C^1 tube system $\{T^{X'}_{i'} : i' \in I_{i_0}\}$ and hence by induction that of $\{T^{X'}_{i'} : X'_{i'} \subset X\}$ with (i), (ii), the first equality in (iii) and (v) for any i_0, i.e., controlled over $\{T^{Y'}_{j'}\}$.

It remains only to consider $X'_{i'}$ in Z, i.e., the case where $X'_{i'}$ is of the form $\{0\} \times Y'_{j'}$ for some j'. Set $\partial I = \{i' : X'_{i'} \subset Z\}$. Obviously, we set

$$\pi^{X'}_{i'}(x) = (0, \pi^{Y'}_{j'} \circ p(x)) \quad \text{for } x \in |T^{X'}_{i'}|,\ i' \in \partial I \text{ and } j' \text{ with } X'_{i'} = \{0\} \times Y'_{j'},$$

where $|T^{X'}_{i'}|$ is a small semialgebraic neighborhood of $X'_{i'}$ in $\mathbf{R}^m \times \mathbf{R}^n$. Then (i) for $i' \in \partial I$ is clear; the first equality in (iii) for $i'_1 \in \partial I$ holds because

$$\begin{aligned} \pi^{X'}_{i'_1} \circ \pi^{X'}_{i'_2}(x) &\overset{\text{definition of } \pi^{X'}_{i'_1}}{=} (0, \pi^{Y'}_{j'_1} \circ p \circ \pi^{X'}_{i'_2}(x)) \overset{\text{(i)}}{=} (0, \pi^{Y'}_{j'_1} \circ \pi^{Y'}_{j'_2} \circ p(x)) \\ &\overset{\text{controlledness of } \{T^{Y'}_{j'}\}}{=} (0, \pi^{Y'}_{j'_1} \circ p(x)) = \pi^{X'}_{i'_1}(x) \quad \text{for } x \in |T^{X'}_{i'_1}| \cap |T^{X'}_{i'_2}|, \end{aligned}$$

where j'_1 and j'_2 are such that $f(X'_{i'_k}) = Y'_{j'_k}$, $k = 1, 2$; (v) for $i'_1 \in \partial I$ is clear, to be precise, for $i'_1 \in \partial I$, i'_2, j'_1 and j'_2 with $X'_{i'_1} \cap (\overline{X'_{i_2}} - X'_{i'_2}) \neq \emptyset$, $Y'_{j'_1} \subset \overline{Y'_{j'_2}} - Y'_{j'_2}$ and $p(X'_{i'_k}) = Y'_{j'_k}$, $k = 1, 2$, the restriction of the map $(\pi^{X'}_{i'_1}, p)$ to $X'_{i'_2} \cap |T^{X'}_{i'_1}|$ is a C^1 submersion into $X'_{i'_1} \times_{(p, \pi^{Y'}_{j'_1})} (Y'_{j'_2} \cap |T^{Y'}_{j'_1}|)$ because $p|_{X'_{i'_1}} : X'_{i'_1} \to Y'_{j'_1}$ is a C^1 diffeomorphism and $p|_{X'_{i'_2}} : X'_{i'_2} \to Y'_{j'_2}$ is a C^1 submersion.

We want to define $\{\rho_{i'}^{X'} : i' \in \partial I\}$ so that $\{T_{i'}^{X'} = (|T_{i'}^{X'}|, \pi_{i'}^{X'}, \rho_{i'}^{X'}) : i' \in \partial I\}$ is a semialgebraic C^1 weak tube system and for each j', $\{T_{i'}^{X'} : f(X'_{i'}) = Y'_{j'}\}$ is controlled. We proceed by double induction. Let $d \geq 0 \in \mathbf{Z}$, and assume $\rho_{i'}^{X'}$ are already defined if $\dim X'_{i'} > d$. We need to construct $\rho_{i'}^{X'}$ for $i' \in \partial I$ with $\dim X'_{i'} = d$. As the problem is local at such $X'_{i'}$, assume there exists only one $i'_0 \in \partial I$ with $\dim X'_{i'_0} = d$. Set $I' = \{i' : X'_{i'_0} \subset \overline{X'_{i'}} - X'_{i'}\}$ and $Y'_{j'_0} = p(X'_{i'_0})$.

For the moment we construct a non-negative semialgebraic C^0 function $\rho_{i'_0,d}^{X'}$ on $|T_{i'_0}^{X'}|$ with zero set $X'_{i'_0}$ which is of class C^1 on $|T_{i'_0}^{X'}| - X'_{i'_0}$ and such that $\{T_{i'_0,d}^{X'}, T_{i'}^{X'} : i' \in I',\, p(X'_{i'}) = Y'_{j'_0}\}$ is controlled, i.e.,

$$\rho_{i'_0,d}^{X'} \circ \pi_{i'}^{X'} = \rho_{i'_0,d}^{X'} \quad \text{on } |T_{i'_0}^{X'}| \cap |T_{i'}^{X'}| \text{ for } i' \in I' \text{ with } p(X'_{i'}) = Y'_{j'_0},$$

where $d = 1 + \#I'$ and $T_{i'_0,d}^{X'} = (|T_{i'_0}^{X'}|, \pi_{i'_0}^{X'}, \rho_{i'_0,d}^{X'})$. (Namely we forget the condition that $\rho_{i'_0,d}^{X'}|_{X'_{i'} \cap \pi_{i'_0}^{X'-1}(x) - X'_{i'_0}}$ is C^1 regular for each x and any $i' \in I'$.) Order elements of I' as $\{i'_1, ..., i'_{d-1}\}$ so that $\dim X'_{i'_1} \leq \cdots \leq \dim X'_{i'_{d-1}}$.

Let $k \in \mathbf{Z}$ with $0 \leq k < d-1$. As the second induction, assume we have a non-negative semialgebraic C^0 function $\rho_{i'_0,k}^{X'}$ defined on $|T_{i'_0}^{X'}| \cap (|T_{i'_1}^{X'}| \cup \cdots \cup |T_{i'_k}^{X'}|)$ such that $\rho_{i'_0,k}^{X'-1}(0) = X'_{i'_0}$, $\rho_{i'_0,k}^{X'}$ is of class C^1 outside of $X'_{i'_0}$ and $\{T_{i'_0,k}^{X'}, T_{i'}^{X'} : i' \in I',\, p(X'_{i'}) = Y'_{j'_0}\}$ is controlled, i.e.,

$$\rho_{i'_0,k}^{X'} \circ \pi_{i'}^{X'} = \rho_{i'_0,k}^{X'} \quad \text{on } |T_{i'_0}^{X'}| \cap (|T_{i'_1}^{X'}| \cup \cdots \cup |T_{i'_k}^{X'}|) \cap |T_{i'}^{X'}| \text{ for } i' \in I' \text{ with } p(X'_{i'}) = Y'_{j'_0},$$

where $T_{i'_0,k}^{X'} = (|T_{i'_0}^{X'}|, \pi_{i'_0}^{X'}, \rho_{i'_0,k}^{X'})$. Then we need to define $\rho_{i'_0,k+1}^{X'}$. Let $\tilde{\rho}_{i'_0,k}^{X'}$ be any non-negative semialgebraic C^0 extension of $\rho_{i'_0,k}^{X'}|_{|T_{i'_0}^{X'}| \cap (|T_{i'_1}^{X'}| \cup \cdots \cup |T_{i'_k}^{X'}|) \cap X'_{i'_{k+1}}}$ to $|T_{i'_0}^{X'}| \cap X'_{i'_{k+1}}$ with zero set $X'_{i'_0}$, let V be an open semialgebraic neighborhood of $X'_{i'_1} \cup \cdots \cup X'_{i'_k}$ in $X'_{i'_1} \cup \cdots \cup X'_{i'_{k+1}}$ whose closure is included in $|T_{i'_1}^{X'}| \cup \cdots \cup |T_{i'_k}^{X'}|$, approximate $\tilde{\rho}_{i'_0,k}^{X'}|_{|T_{i'_0}^{X'}| \cap X'_{i'_{k+1}} - V}$ by a non-negative semialgebraic C^0 function $\tilde{\tilde{\rho}}_{i'_0,k}^{X'}$ in the uniform C^0 topology so that $\tilde{\tilde{\rho}}_{i'_0,k}^{X'-1}(0) = X'_{i'_0}$, and $\tilde{\tilde{\rho}}_{i'_0,k}^{X'}$ is of class C^1 outside of $X'_{i'_0}$ (Theorem II.4.1, [3]), let ξ be a semialgebraic C^1 function on $|T_{i'_0}^{X'}| \cap X'_{i'_{k+1}}$ such that $0 \leq \xi \leq 1$, $\xi = 0$ on $|T_{i'_0}^{X'}| \cap X'_{i'_{k+1}} \cap V$ and $\xi = 1$ on $|T_{i'_0}^{X'}| \cap X'_{i'_{k+1}} - |T_{i'_1}^{X'}| - \cdots |T_{i'_k}^{X'}|$, and set

$$\hat{\rho}_{i'_0,k}^{X'}(x) = \xi(x)\tilde{\tilde{\rho}}_{i'_0,k}^{X'}(x) + (1 - \xi(x))\rho_{i'_0,k}^{X'}(x) \quad \text{for } x \in |T_{i'_0}^{X'}| \cap X'_{i'_{k+1}}.$$

Then $\hat{\rho}_{i'_0,k}^{X'}$ is a non-negative semialgebraic C^0 extension of $\rho_{i'_0,k}^{X'}|_{|T_{i'_0}^{X'}| \cap V \cap X'_{i'_{k+1}}}$ to $|T_{i'_0}^{X'}| \cap X'_{i'_{k+1}}$ with zero set $X'_{i'_0}$ and of class C^1 outside of $X'_{i'_0}$. If $p(X'_{i'_{k+1}}) \neq Y'_{j'_0}$, we continue to extend $\hat{\rho}_{i'_0,k}^{X'}$ to the required $\hat{\rho}_{i'_0,k+1}^{X'} : |T_{i'_0}^{X'}| \cap (|T_{i'_1}^{X'}| \cup \cdots \cup |T_{i'_{k+1}}^{X'}|) \to \mathbf{R}$ shrinking $|T_{i'_1}^{X'}|, ..., |T_{i'_k}^{X'}|$ and using a partition of unity in the same way so that $\rho_{i'_0,k+1}^{X'} = \rho_{i'_0,k}^{X'}$ on

$|T^{X\prime}_{i'_0}| \cap (|T^{X\prime}_{i'_1}| \cup \cdots \cup |T^{X\prime}_{i'_k}|)$. Otherwise, set

$$\rho^{X\prime}_{i'_0,k+1} = \begin{cases} \rho^{X\prime}_{i'_0,k} & \text{on } |T^{X\prime}_{i'_0}| \cap (|T^{X\prime}_{i'_1}| \cup \cdots \cup |T^{X\prime}_{i'_k}|) \\ \hat{\rho}^{X\prime}_{i'_0,k} \circ \pi^{X\prime}_{i'_0,k+1} & \text{on } |T^{X\prime}_{i'_0}| \cap |T^{X\prime}_{i'_{k+1}}|, \end{cases}$$

which is well-defined because

$$\begin{aligned} &\hat{\rho}^{X\prime}_{i'_0,k} \circ \pi^{X\prime}_{i'_0,k+1} = \rho^{X\prime}_{i'_0,k} \circ \pi^{X\prime}_{i'_0,k+1} \quad \text{by definition of } \hat{\rho}^{X\prime}_{i'_0,k} \\ &= \rho^{X\prime}_{i'_0,k} \quad \text{by controlledness of } \{T^{X\prime}_{i'_0,k}, T^{X\prime}_{i'} : i' \in I',\ p(X'_{i'}) = Y'_{j'_0}\} \\ &\qquad \text{on } |T^{X\prime}_{i'_0}| \cap (|T^{X\prime}_{i'_1}| \cup \cdots \cup |T^{X\prime}_{i'_k}|) \cap |T^{X\prime}_{i'_{k+1}}| \text{ for shrunk } |T^{X\prime}_{i'_1}|, ..., |T^{X\prime}_{i'_k}|. \end{aligned}$$

Then clearly $\rho^{X\prime-1}_{i'_0,k+1}(0) = X'_{i'_0}$, $\rho^{X\prime}_{i'_0,k+1}$ is of class C^1 outside of $X'_{i'_0}$ and

$\rho^{X\prime}_{i'_0,k+1} \circ \pi^{X\prime}_{i'} = \rho^{X\prime}_{i'_0,k+1}$ on $|T^{X\prime}_{i'_0}| \cap (|T^{X\prime}_{i'_1}| \cup \cdots \cup |T^{X\prime}_{i'_{k+1}}|) \cap |T^{X\prime}_{i'}|$ for $i' \in I'$ with $p(X'_{i'}) = Y'_{j'_0}$

as follows. It suffices to consider only the case where $\overline{X'_{i'}} - X'_{i'} \supset X'_{i'_{k+1}}$ and $p(X'_{i'}) = p(X'_{i'_{k+1}}) = Y'_{j'_0}$ and the equation on $|T^{X\prime}_{i'_0}| \cap |T^{X\prime}_{i'_{k+1}}| \cap |T^{X\prime}_{i'}|$. We have

$$\begin{aligned} \rho^{X\prime}_{i'_0,k+1} \circ \pi^{X\prime}_{i'} &= \hat{\rho}^{X\prime}_{i'_0,k} \circ \pi^{X\prime}_{i'_{k+1}} \circ \pi^{X\prime}_{i'} \quad \text{by definition of } \rho^{X\prime}_{i'_0,k+1} \\ &= \hat{\rho}^{X\prime}_{i'_0,k} \circ \pi^{X\prime}_{i'_{k+1}} \quad \text{by the first equation in (iii)} \\ &= \rho^{X\prime}_{i'_0,k+1} \quad \text{by definition of } \rho^{X\prime}_{i'_0,k+1} \quad \text{on } |T^{X\prime}_{i'_0}| \cap |T^{X\prime}_{i'_{k+1}}| \cap |T^{X\prime}_{i'}|. \end{aligned}$$

Thus by the second induction we obtain $\rho^{X\prime}_{i'_0,d-1} : |T^{X\prime}_{i'_0}| \cap (|T^{X\prime}_{i'_1}| \cup \cdots \cup |T^{X\prime}_{i',d-1}|) \to \mathbf{R}$.

It remains only to extend $\rho^{X\prime}_{i'_0,d-1}$ to a non-negative semialgebraic C^0 function $\rho^{X\prime}_{i'_0,d}$ on $|T^{X\prime}_{i'_0}|$ with zero set $X'_{i'_0}$ and of class C^1 outside of $X'_{i'_0}$. However we have already carried out such a sort of extension by using a partition of unity ξ.

We need to solve the problem of C^1 regularity of $\rho^{X\prime}_{i'_0,d}|_{X'_{i'} \cap \pi^{X\prime-1}_{i'_0}(x) - X'_{i'_0}}$. For each $x \in X'_{i'_0}$, the restriction of $\rho^{X\prime}_{i'_0,d}$ to $X'_{i'} \cap \pi^{X\prime-1}_{i'_0}(x) \cap \rho^{X\prime-1}_{i'_0,d}((0,\ \delta_x))$ is C^1 regular for some $\delta_x > 0 \in \mathbf{R}$ and any $i' \in I'$. Here we can choose δ_x so that the function $X'_{i'_0} \ni x \to \delta_x \in \mathbf{R}$ is semialgebraic (but not necessarily continuous). Then there exists a semialgebraic closed subset $X''_{i'_0}$ of $X'_{i'_0}$ of smaller dimension such that each point x in $X'_{i'_0} - X''_{i'_0}$ has a neighborhood in $X'_{i'_0}$ where δ_x is larger than a positive number. Hence if we replace $X'_{i'_0}$ with $X'_{i'_0} - X''_{i'_0}$, i.e., $Y'_{j'_0}$ with $Y'_{j'_0} - p(X''_{i'_0})$ and shrink $|T^{X\prime}_{i'_0}|$ then the C^1 regularity holds. Thus we obtain the required $\rho^{X\prime}_{i'_0}$ though $X'_{i'_0}$ is shrunk to $X'_{i'_0} - X''_{i'_0}$.

The shrinking is admissible as follows. Substratify $\{Y'_{j'} \cap p(\overline{X''_{i'_0}}),\ Y'_{j'} - p(\overline{X''_{i'_0}})\}$ to a Whitney semialgebraic C^1 stratification $\{Y''_{j''}\}$ such that $\{Y'_{j'} - p(\overline{X''_{i'_0}})\} = \{Y''_{j''} - p(\overline{X''_{i'_0}})\}$, set $\{X''_{i''}\} = \{X_{i,j} \cap p^{-1}(Y''_{j''}),\ Z \cap \{0\} \times Y''_{j''}\}$, which implies $\{X'_{i'} - p^{-1}(p(\overline{X''_{i'_0}}))\} = \{X''_{i''} - p^{-1}(p(\overline{X''_{i'_0}}))\}$, and repeat all the above arguments to $\overline{f} : \{X''_{i''}\} \to \{Y''_{j''}\}$. Then we obtain a semialgebraic C^1 tube system $\{T^{Y\prime\prime}_{j''}\}$ for $\{Y''_{j''}\}$ and a semialgebraic C^1 tube system $\{T^{X\prime\prime}_{i''} : X''_{i''} \subset X\}$ for $\{X''_{i''} \subset X\}$ controlled over $\{T^{Y\prime\prime}_{j''}\}$ such that $\{T^{Y\prime\prime}_{j''} : Y''_{j''} \cap p(\overline{X''_{i'_0}}) =$

$\emptyset\}$ and $\{T^{X''}_{i''} : X''_{i''} \subset X,\ X''_{i''} \cap p^{-1}(p(\overline{X''_{i'_0}})) = \emptyset\}$ are equal to $\{T^{Y'}_{j'}|_{|T^{Y'}_{j'}|-\pi^{Y'-1}_{j'}(p(\overline{X''_{i'_0}}))}\}$ and $\{T^{X'}_{i'}|_{|T^{X'}_{i'}|-\pi^{X'-1}_{i'}(p^{-1}(p(\overline{X''_{i'_0}})))}\}$, respectively, by (iv) and (ix), where the domains of the latter two tube systems are shrunk. Moreover we continue construction of $\rho^{X''}_{i''}$ for $X'' \subset Z$. Since $\{X''_{i''} \subset Z : \dim X''_{i''} > d\} = \{X'_{i'} \subset Z : \dim X'_{i'} > d\}$ and $\{X''_{i''} \subset Z : \dim X''_{i''} = d\} = \{X'_{i'_0} - X''_{i'_0}\}$ we choose $\rho^{X'}_{i'}$ as $\rho^{X''}_{i''}$ for $X''_{i''} \subset Z$ with $\dim X''_{i''} > d$ and $\rho^{X'}_{i'_0}|_{|T^{X''}_{i''}|}$ as $\rho^{X''}_{i''}$ for $X''_{i''} \subset Z$ with $\dim X''_{i''} = d$. Hence we can assume $X''_{i'_0} = \emptyset$ from the beginning, which completes the construction of $\rho^{X'}_{i'_0}$ and hence of the required $\{\rho^{X'}_{i'} : i' \in \partial I\}$ by induction.

Thus $\overline{f} : \{X'_{i'}\} \to \{Y'_{j'}\}$, $\{T^{X'}_{i'}\}$ and $\{T^{Y'}_{j'}\}$ satisfy the conditions in theorem 2.2. Hence theorem 1.2 follows. $\square$

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